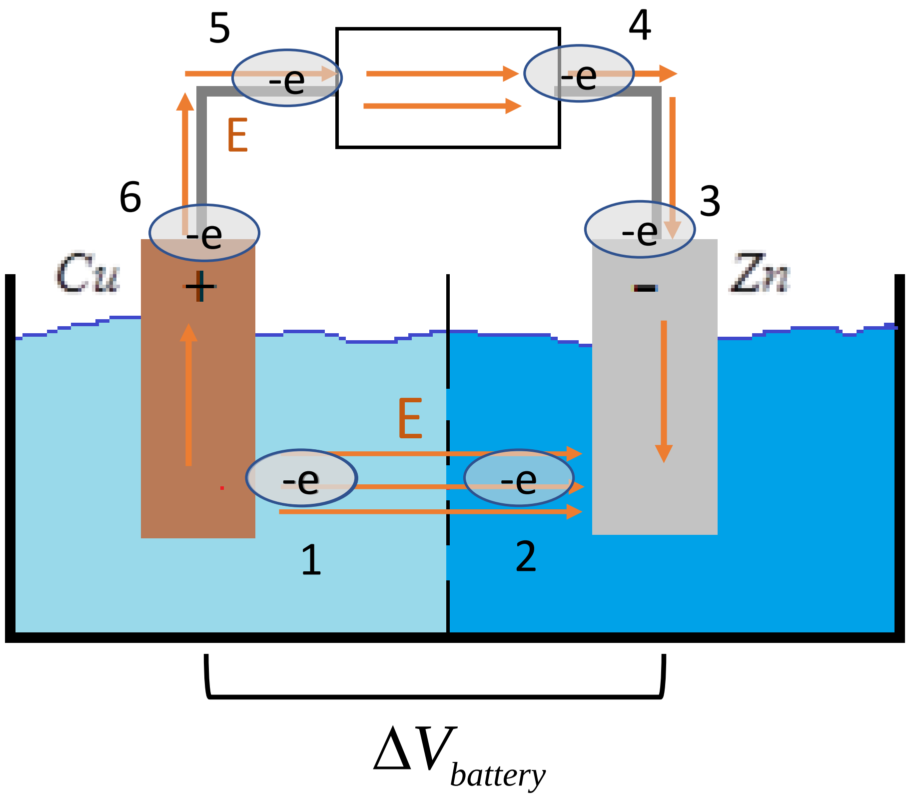
B.3 Resistors

So far we've discussed ways to deliver energy to the electrons traversing a circuit, either continuously via batteries, or somewhat ephemerally via capacitors. And now we've gotta consider what the electrons are doing with that energy, and more generally, some of the details of their flow through the wires in the circuit.



upshot: battery loses energy $e\Delta V_{battery}$
atoms gain thermal energy $e\Delta V_{battery}$

1 to 2: battery transfers chemical potential energy to electron, giving it $\Delta PE_E = e\Delta V_{battery}$, and enabling its transport across the internal field.

2 to 3: electron is pulled by field in cathode to the terminal. It loses PE_E, gains KE, and simultaneously delivers that KE to the cathode atoms via collisions.

3 to 4: electron is pulled by field in wire from terminal to device/resistor. It loses PE_E, gains KE, and simultaneously delivers that KE it to the wire atoms via collisions.

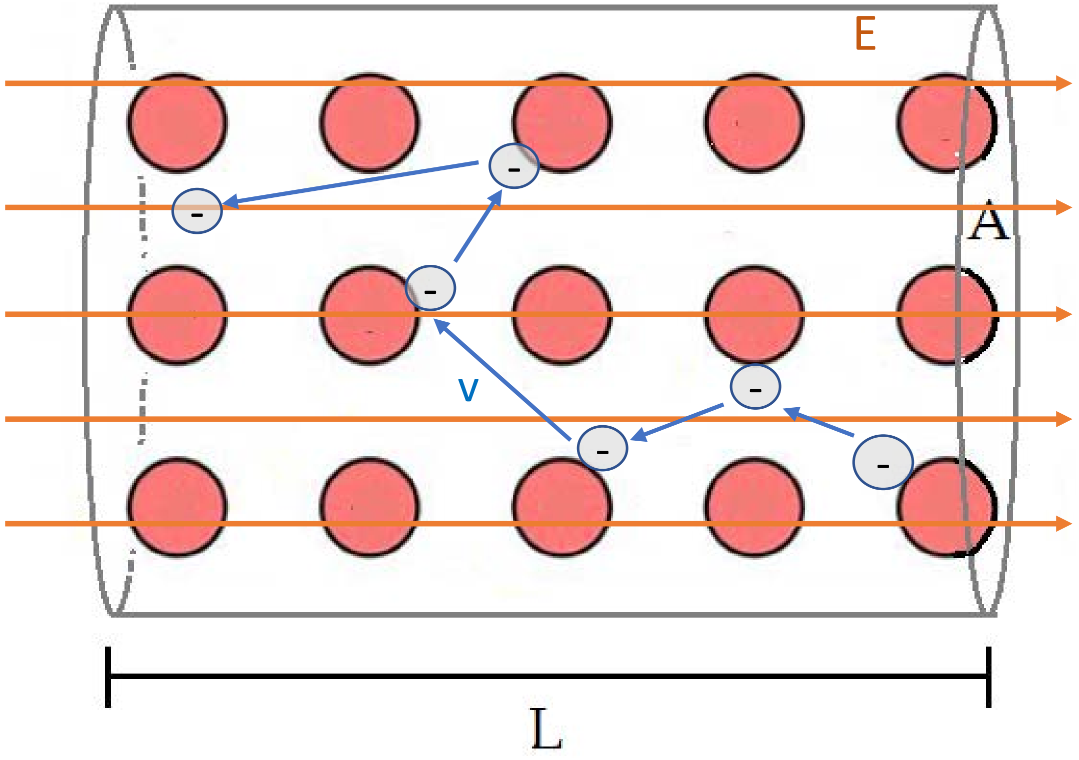
4 to 5: electron is pulled by field across device/resistor by field, which is typically much stronger than in the wire. The electron loses PE_E, gains KE, and simultaneously delivers that KE to the device/resistor atoms via collisions.

5 to 6: electron is pulled by field in wire from device/resistor to terminal. It loses PE_E, gains KE, and simultaneously delivers that KE to the wire atoms via collisions.

6 to 1: electron is pulled by field in anode from terminal. It loses PE_E, gains KE, and simultaneously delivers that KE to the cathode atoms via collisions.

B.3 Resistors

Now we would like to consider more details about how quickly the electrons will flow through the circuit.
Let's concentrate on the motion of just one electron through the resistor:



It will generally accelerate against the direction of the field of course, but it will never pick up much speed because of the frequent collisions it makes with the atoms in the resistor.

So it will basically drift along at a constant rate (speed), v.
This speed is given by:

$$v = v_0 + at$$

average time in between collisions is called τ. This parameter must be looked up in table (or can calculate via other means)

average initial velocity after a collision is zero.

acceleration given by:

$$a = \frac{F}{m} = \frac{eE}{m}$$

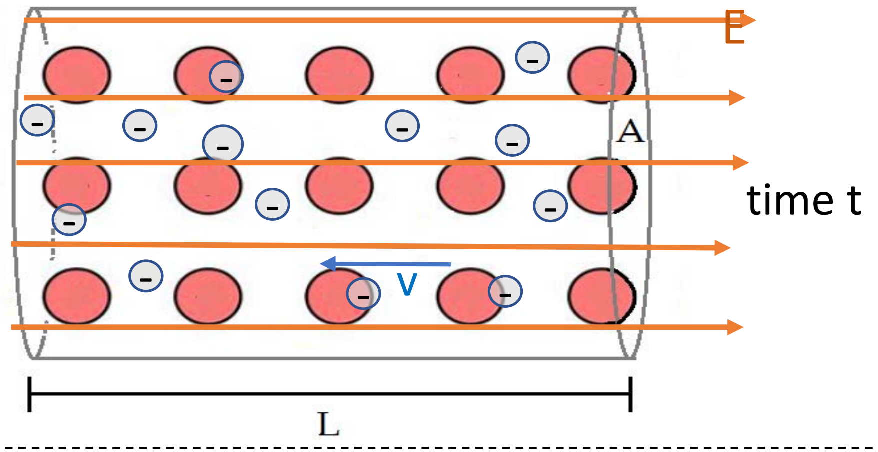
So $v = 0 + \frac{eE}{m} \cdot \tau \longrightarrow \boxed{v = \frac{eE}{m}\tau}$

B.3 Resistors

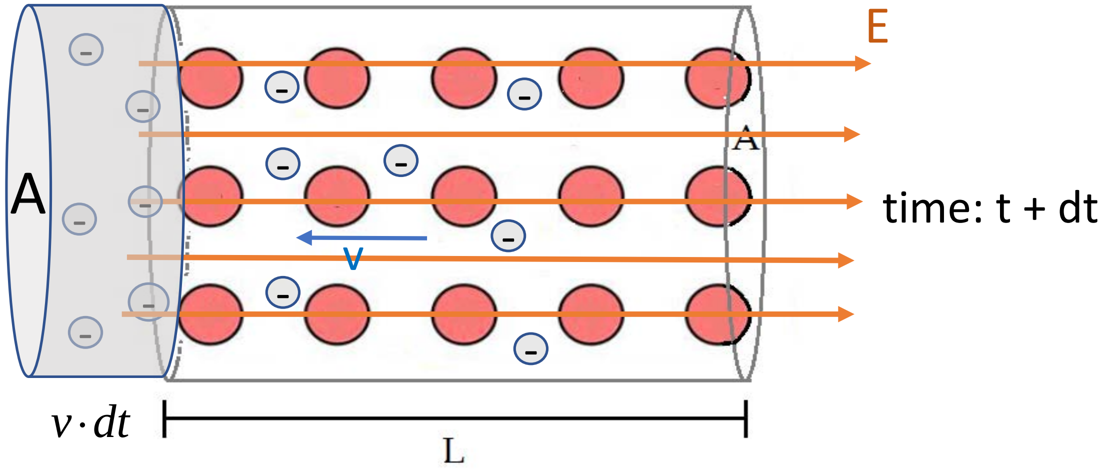
Now we'd like to relate the speed of these electrons to the 'current'. So the current of electrons through a wire is defined to be:

$$I = \frac{dq}{dt} = \frac{\text{amount of charge, } dq\text{, that flows past a point on wire in time } dt}{dt}$$

units: Coulombs/second = Ampere (A)



Consider a bunch of electrons in a wire flowing left with velocity v.



A time dt later, an amount dq will have flowed past the wire. And this amount would be, in terms of their volume dV:

$$dq = nedV \qquad n = \#\text{ electrons per unit volume}$$

$$= n(Avdt)$$

And so the current would be:

$$I = \frac{dq}{dt} = \frac{neAvdt}{dt} = neAv \longrightarrow \boxed{I = nevA}$$

B.3 Resistors

Parenthetically, it is convenient to do a little work on the electron density formula….

$$
\begin{aligned}
n &= \frac{\#\text{charge carriers}}{\text{volume}} \\
&= \frac{\#\text{ charge carriers}}{\text{atom}}\frac{\#\text{atoms}}{\text{volume}} \\
&= \frac{\#\text{ charge carriers}}{\text{atom}}\frac{\#\text{atoms}}{\text{mass}}\frac{\text{mass}}{\text{volume}} \\
&= \frac{\#\text{ charge carriers}}{\text{atom}}\frac{\#\text{atoms}}{\text{mol}}\frac{\text{mol}}{\text{mass}}\frac{\text{mass}}{\text{volume}} \\
&= \frac{\#\text{ charge carriers}}{\text{atom}}\frac{\#\text{atoms}}{\text{mol}}\frac{1}{\frac{\text{mass}}{\text{mol}}}\text{density} \\
&= \frac{\tilde{n} N_A \rho_{mass}}{m_{molar}}
\end{aligned}
$$

And also make some definitions

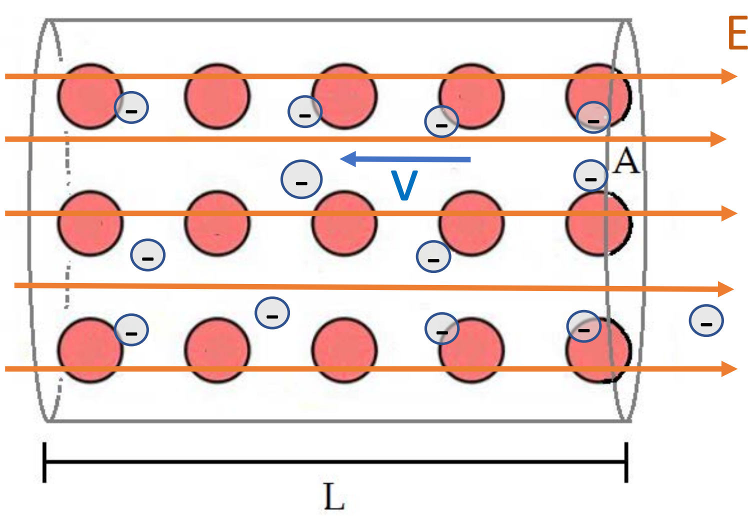
$$
\begin{aligned}
I &= nevA \\
&= \lambda v \\
&= \sigma v l \\
&= \rho v A
\end{aligned}
$$

$$\lambda = neA$$

$$K = \sigma e v$$

B.3 Resistors

And now we're in position to get Ohm's law, which relates the current flowing through a resistor to the potential difference across it:



$$I = nevA = ne\left(\frac{eE\tau}{m}\right)A = \frac{ne^2\tau}{m}EA = \frac{ne^2\tau}{m}\frac{\Delta V}{L}A = \frac{\Delta V}{\left(\frac{m}{ne^2\tau}\right)\frac{L}{A}}$$

$$\text{resistivity:}\quad \rho = \frac{m}{ne^2\tau} \qquad \text{units} = \frac{\text{kg}}{\frac{1}{\text{m}^3}\cdot\text{C}^2\cdot\text{s}} = \frac{\text{J/C}}{\text{C/s}}\cdot\text{m} = \frac{V}{A}\cdot\text{m} = \Omega\cdot\text{m}$$

$$\frac{V}{A} \equiv \text{Ohm } (\Omega)$$

$$\text{Resistance: } R = \left(\frac{m}{ne^2\tau}\right)\frac{L}{A} = \rho\frac{L}{A} \qquad \text{units} = \Omega\cdot\text{m}\cdot\frac{\text{m}}{\text{m}^2} = \Omega$$

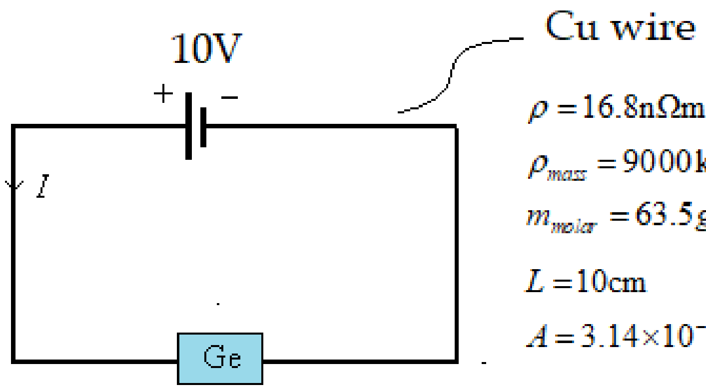
$$I = \frac{\Delta V}{R} \qquad \text{Ohm's Law}$$

Rate of energy dissipation by electrons is:

$$P = \text{rate of dissipation of KE}$$
$$= \text{rate of loss of } PE_E$$
$$= I\Delta V$$

$$P = I^2R$$

B.3 Resistors



$\rho = 16.8\,\mathrm{n\Omega m}$

$\rho_{mass} = 9000\,\mathrm{kg/m}$

$m_{molar} = 63.5\,g$

$L = 10\,\mathrm{cm}$

$A = 3.14\times10^{-6}\,\mathrm{m}^2$

$\rho = 0.46\,\Omega\mathrm{m}$

$\rho_{mass} = 5500\,\mathrm{kg/m}^3$

$m_{molar} = 72.6\,\mathrm{g}$

$L = 1\mathrm{cm}$

$A = 3.14\times10^{-5}\,\mathrm{m}^2$

Suppose we hook a battery up to a Germanium resistor via two identical copper wires. The specifications of the wire, and the resistor are given.

(a) What's the resistance of the Ge resistor and of the Cu wires?

$$R_{Cu} = \frac{L}{A}\rho_{Cu} = \frac{0.1\mathrm{m}}{3.14\times10^{-6}\,\mathrm{m}^2}(1.68\times10^{-8}\,\Omega\cdot\mathrm{m}) = 5.34\times10^{-4}\,\Omega$$

$$R_{Ge} = \frac{L}{A}\rho_{Ge} = \frac{0.01\mathrm{m}}{3.14\times10^{-5}\,\mathrm{m}^2}(0.46\,\Omega\cdot\mathrm{m}) = 146\,\Omega$$

(b) What's the total resistance of the circuit, and what's the current running through it?

$$R_{eq.} = 5.34\times10^{-4}\,\Omega + 146\Omega + 5.34\times10^{-4}\,\Omega \approx 146\,\Omega$$

$$I = \frac{\Delta V}{R_{eq.}} = \frac{10\,\mathrm{V}}{146\,\Omega} = 0.068A$$

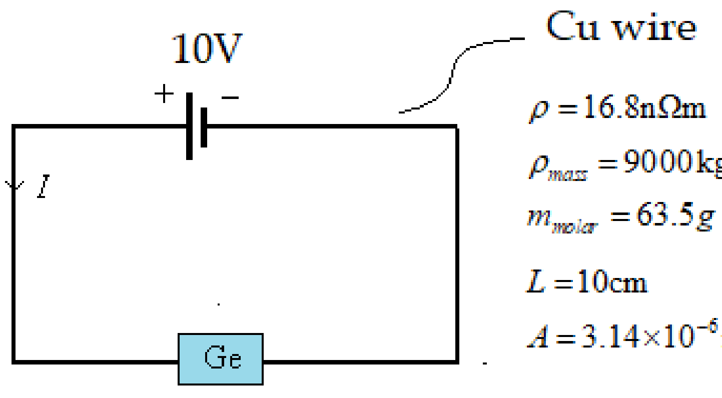
(c) What's the power delivered by the battery, and dissipated by the resistor and wires?

$$P_b = I\Delta V_b = (0.068\mathrm{A})(10\mathrm{V}) = 0.68\,\mathrm{W}$$

$$P_{Cu} = I^2R = (0.068)^2(5.34\times10^{-4}) = 2.47\times10^{-6}\,\mathrm{W}$$

$$P_{Ge} = I^2R = (0.068)^2(146) \approx 0.68\,\mathrm{W}$$

B.3 Resistors



Cu wire:

$\rho = 16.8\,\text{n}\Omega\text{m}$

$\rho_{mass} = 9000\,\text{kg/m}$

$m_{molar} = 63.5\,\text{g}$

$L = 10\,\text{cm}$

$A = 3.14\times10^{-6}\,\text{m}^2$

Ge:

$\rho = 0.46\,\Omega\text{m}$

$\rho_{mass} = 5500\,\text{kg/m}^3$

$m_{molar} = 72.6\,\text{g}$

$L = 1\,\text{cm}$

$A = 3.14\times10^{-5}\,\text{m}^2$

(d) What's the speed of the electrons in the Cu wire?

Speed is v = I/neA. And so we'll need to get n.

$$n_{Cu} = \frac{\tilde{n}N_A\rho_{mass}}{m_{molar}} = \frac{(1)(6.022\times10^{23})(9000)}{0.0635} = 8.5\times10^{28}\ \text{electrons/m}^3$$

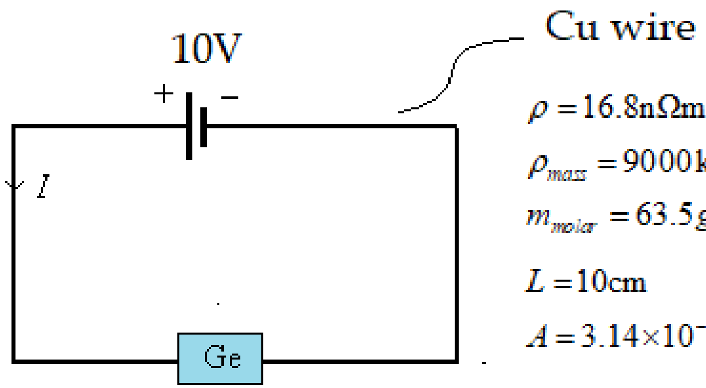
And so the speed is:

$$v_{Cu} = \frac{I}{n_{Cu}eA_{Cu}} = \frac{0.068}{(8.5\times10^{28})(1.6\times10^{-19})(3.14\times10^{-6})} = 1.6\,\mu\text{m/s}$$

(e) What's the mean collision time in the Cu wire?

$$\rho = \frac{m}{ne^2\tau} \qquad \tau = \frac{m}{\rho ne^2} \qquad \tau = \frac{(9.11\times10^{-31})}{(1.68\times10^{-8})(8.5\times10^{28})(1.6\times10^{-19})^2} = 2.5\times10^{-8}\,\text{s} = 25\,\text{ns}$$

B.3 Resistors



$\rho = 16.8\,\mathrm{n\Omega m}$

$\rho_{mass} = 9000\,\mathrm{kg/m}$

$m_{molar} = 63.5\,g$

$L = 10\mathrm{cm}$

$A = 3.14\times10^{-6}\mathrm{m}^2$

$\rho = 0.46\,\Omega\mathrm{m}$

$\rho_{mass} = 5500\,\mathrm{kg/m}^3$

$m_{molar} = 72.6\,\mathrm{g}$

$L = 1\mathrm{cm}$

$A = 3.14\times10^{-5}\mathrm{m}^2$

(f) What’s the strength of the electric field in the Cu wire? in the Ge resistor?

$$\Delta V_{Cu} = IR = (0.068)(5.34\times10^{-4}) = 3.61\times10^{-5}\ \mathrm{V}$$

$$E_{Cu} = \frac{|\Delta V_{Cu}|}{\Delta x} = \frac{3.61\times10^{-5}\,V}{0.1\mathrm{m}} = 3.61\times10^{-4}\,\frac{\mathrm{V}}{\mathrm{m}}$$

$$\Delta V_{Ge} = IR_{Ge} = (0.068)(146) \approx 10\ \mathrm{V}$$

$$E_{Ge} = \frac{|\Delta V_{Ge}|}{\Delta x} = \frac{10\,V}{0.01\mathrm{m}} = 1000\,\frac{\mathrm{V}}{\mathrm{m}}$$

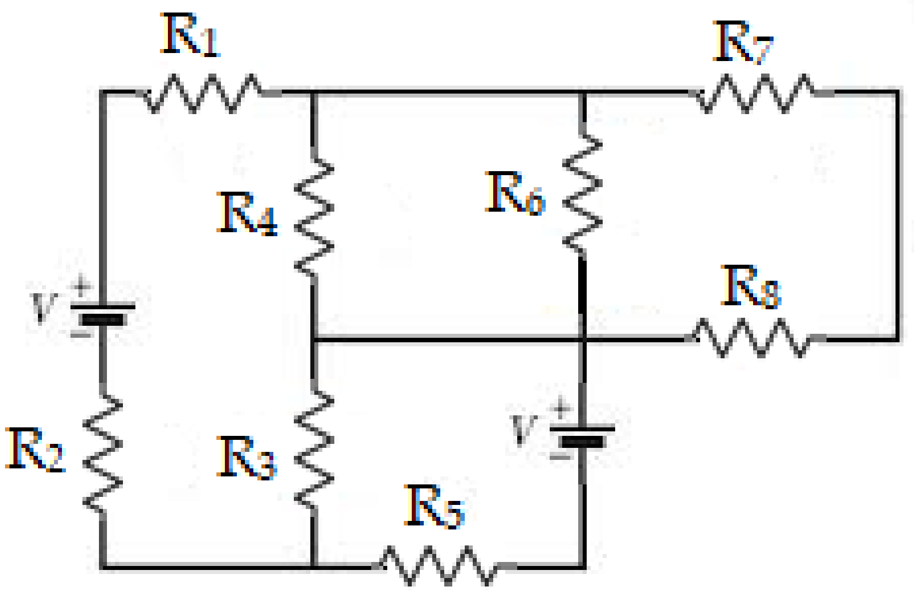
B.3 Network of Resistors

Now we'll generalize to a network of resistors.
Our goal will be to determine the power and energy dissipated through any resistor.

Just as it was with capacitor networks, there are two methods to address this question:

(a) Method of equivalent resistance.
(b) Kirchoff's laws.

The first is faster, when applicable, and the latter more general. We'll consider the first, first.



B.3 Network of Resistors

Method of equivalent resistance aims to simplify a network of resistors by systematically replacing certain combinations of resistors with a single 'equivalent' resistor. The two resistor combinations amenable to this method are *series*, and *parallel*. Two resistors are connected 'in series' or 'in parallel' if they satisfy the following criteria:

Series: Can draw a path along circuit, from one resistor to the other, w/o crossing a junction.

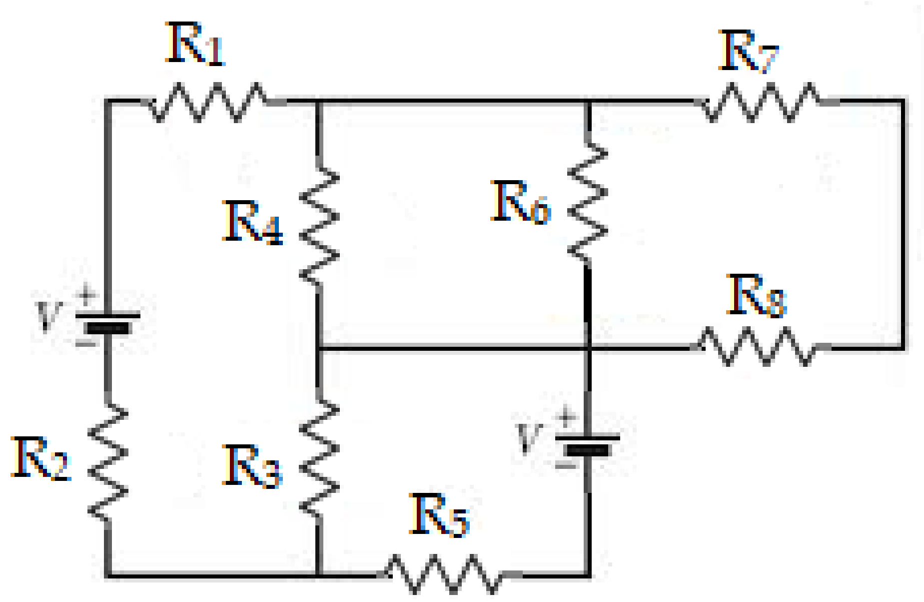
Parallel: Can draw a closed loop along the circuit, from one resistor to the other, w/o crossing another circuit element (i.e., battery, capacitor, resistor, etc.)

Which resistors are in parallel?

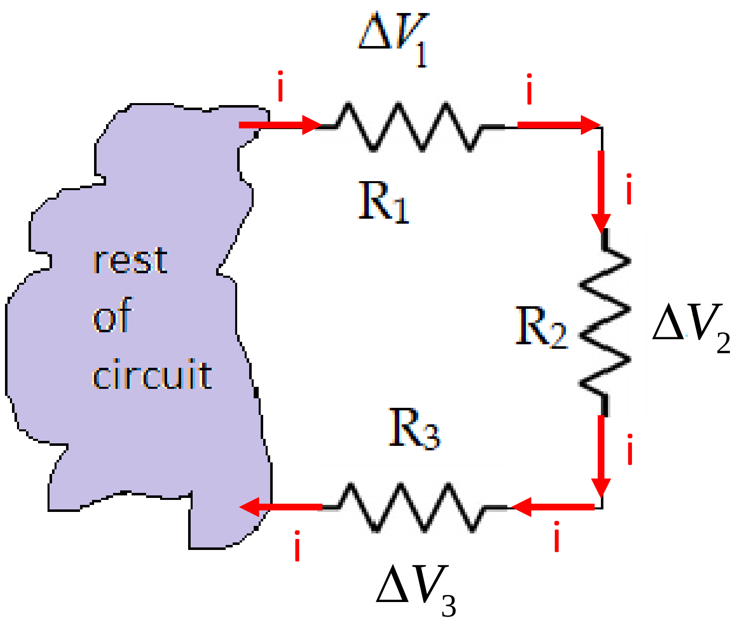
R_4 and R_6

Which resistors are in series?

R_1 and R_2
R_7 and R_8



B.3 Network of Resistors



If current starts out as i, what is the current everywhere else?

Currents must be the same b/c of charge conservation.

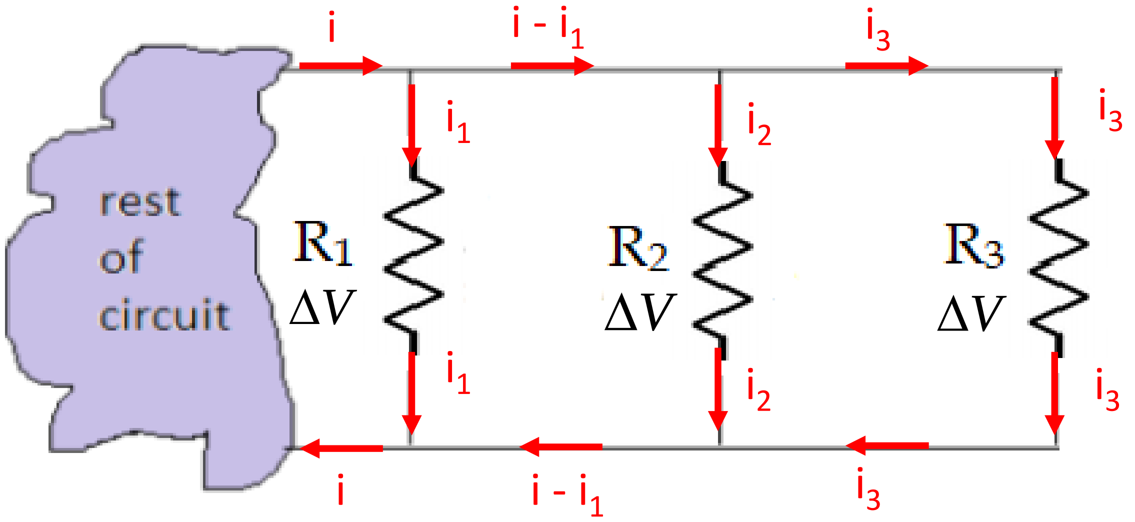
Fun Fact: Resistors in series
all carry the same current

How do currents relate in a parallel connection?

How do voltages relate?

Voltages are same since ΔV must be zero around a closed loop.

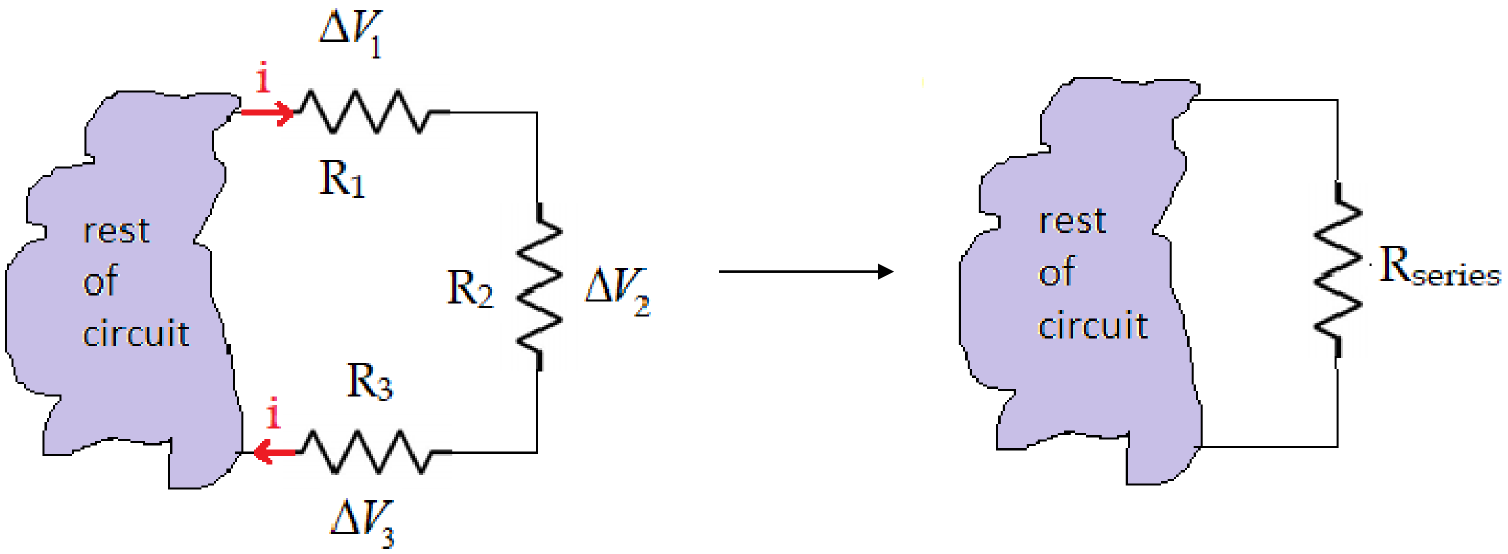
Fun Fact: Resistors in parallel
all carry the same voltage difference



B.3 Network of Resistors

Series equivalent resistance

Now we need to figure out what the equivalent resistance of a bunch of resistors connected in series, is. The *equivalent resistance* is defined as the single resistor that would draw the same current under same voltage, as the series combination.



$$\Delta V_{series} = \Delta V_1 + \Delta V_2 + \Delta V_3$$

$$i_{series} = i$$

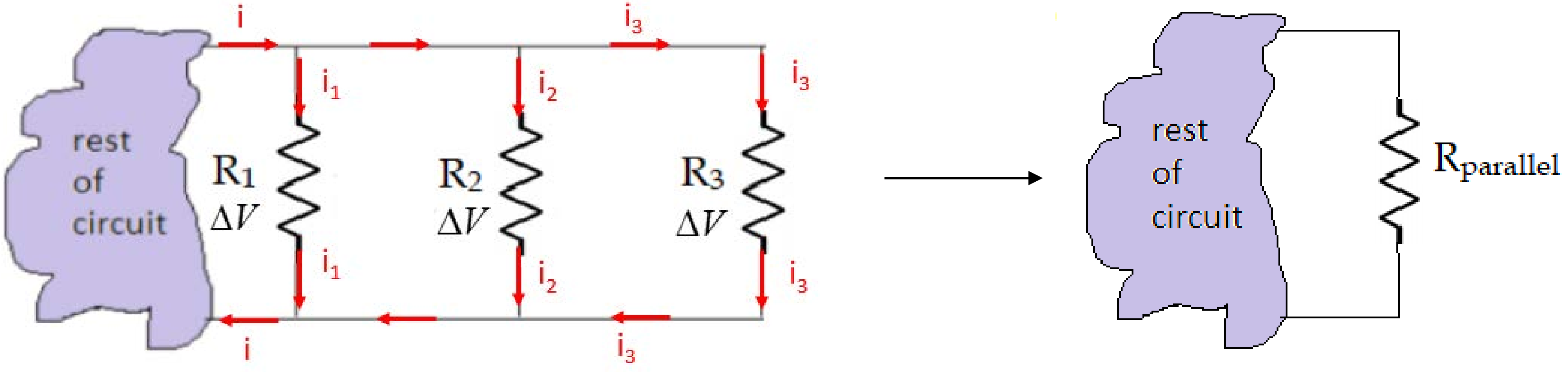
$$\left.\begin{aligned} R_{series} &= \frac{\Delta V_{series}}{i_{series}} = \frac{\Delta V_1 + \Delta V_2 + \Delta V_3}{i} \\ &= \frac{\Delta V_1}{i} + \frac{\Delta V_2}{i} + \frac{\Delta V_3}{i} = R_1 + R_2 + R_3 \end{aligned}\right\}$$

$$R_{series} = R_1 + R_2 + R_3 + \ldots + R_n$$

B.3 Network of Resistors

Parallel equivalent resistance

Now we need to figure out what the equivalent resistance of a bunch of resistors connected in parallel, is. The *equivalent resistance* is defined as the single resistor that would draw the same current under same voltage, as the parallel combination.



$$\Delta V_{parallel} = \Delta V$$

$$i_{parallel} = i_1 + i_2 + i_3$$

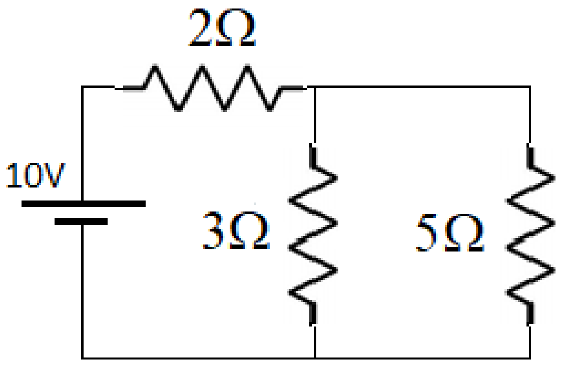
$$R_{parallel} = \frac{\Delta V_{parallel}}{i_{parallel}} = \frac{\Delta V}{i_1 + i_2 + i_3} = \frac{1}{i_1/\Delta V + i_2/\Delta V + i_3/\Delta V}$$

$$= \frac{1}{1/R_1 + 1/R_2 + 1/R_3}$$

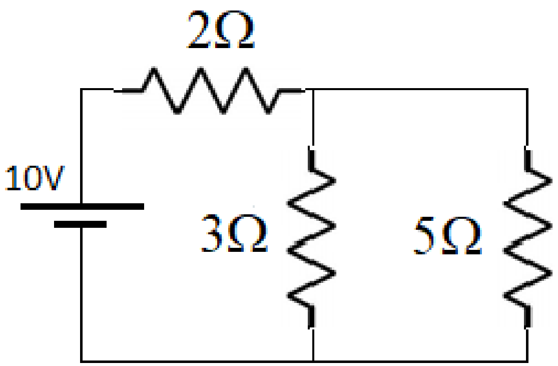
$$R_{parallel} = (R_1^{-1} + R_2^{-1} + R_3^{-1} + \dots + R_n^{-1})^{-1}$$

B.3 Network of Resistors

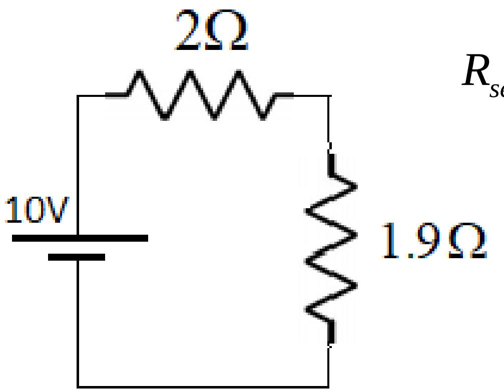
Consider the following amazing circuit that looks just like a previous amazing circuit. What's the current drawn from the battery? What's the current through each resistor? What's the power supplied by the battery? What's the power dissipated through each resistor? If these are lightbulbs, rank them in order of brightness.



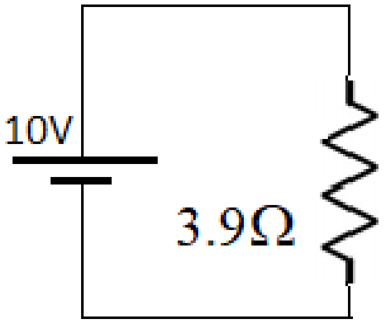
General procedure is to ultimately reduce it to one resistor.
We get:



$$R_{parallel} = (3^{-1} + 5^{-1})^{-1}$$
$$= 1.9\Omega$$



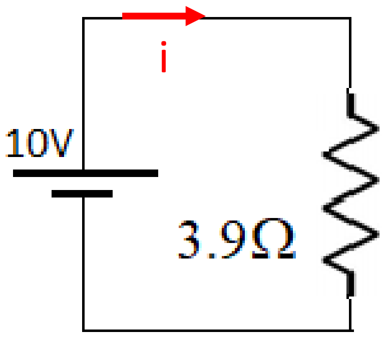
$$R_{series} = 2 + 1.9$$
$$= 3.9\Omega$$



B.3 Network of Resistors

This is the power that the battery supplies to the device

$$P_b = I\Delta V_b = (2.6)(10)$$
$$= 26\text{W}$$



$$i = \frac{\Delta V}{R} = \frac{10\text{V}}{3.9\Omega} = 2.6\text{A}$$

This is the current that the battery supplies to the circuit.

And now let's work backwards to get the currents through the individual resistors…

$i = 2.6\,\text{A}$

$\Delta V = (2.6\text{A})(2\Omega)$

$= 5.2\text{V}$

$i = 2.6\,\text{A}$

$\Delta V = (2.6\text{A})(1.9\Omega)$

$= 4.8\text{V}$

$i = 2.6\,\text{A}$

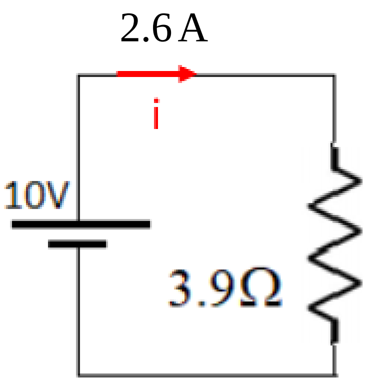
$\Delta V = 5.2\text{V}$

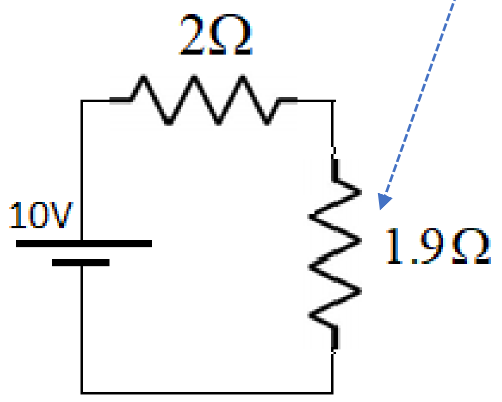
$\Delta V = 4.8\text{V}$

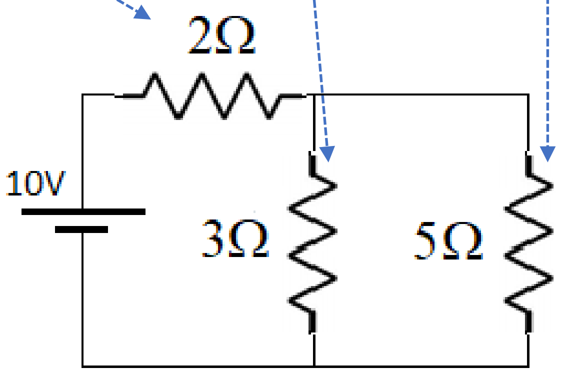
$i = \dfrac{4.8\text{V}}{3\Omega} = 1.6\,\text{A}$

$\Delta V = 4.8\text{V}$

$i = \dfrac{4.8\text{V}}{5\Omega} = 1\text{A}$







B.3 Network of Resistors

Last we wanted to get the power dissipated by each resistor…

$i = 2.6\,\mathrm{A}$

$\Delta V = 5.2\mathrm{V}$

$P = i^2 R = (2.6\mathrm{A})^2 (2\Omega) = 13\mathrm{W}$

brightest

$\Delta V = 4.8\mathrm{V}$

$i = 1.6\,\mathrm{A}$

$P = i^2 R = (1.6\mathrm{A})^2 (3\Omega) = 8\mathrm{W}$

bright

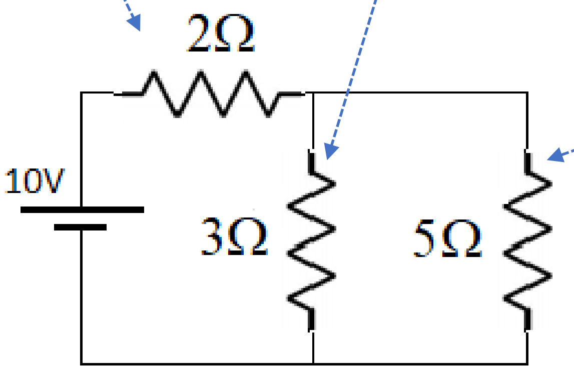
$\Delta V = 4.8\mathrm{V}$

$i = 1\mathrm{A}$

$P = i^2 R = (1\mathrm{A})^2 (5\Omega) = 5\mathrm{W}$

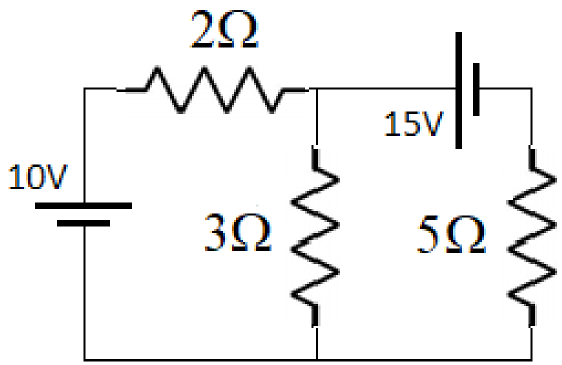
least bright

$P_b = 26\mathrm{W}$

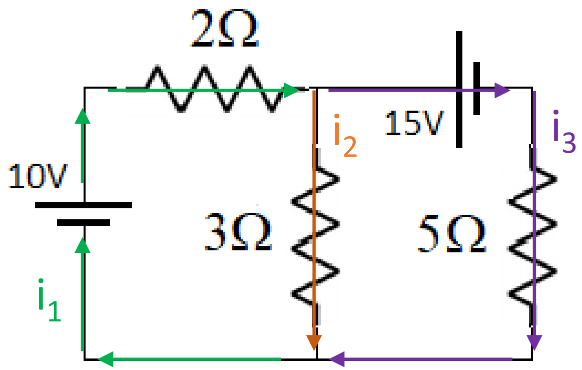


B.3 Network of Resistors

Just when you thought the circuit couldn't get any more amazing, this happened.   Now how much power is being supplied by the circuit? How much is being dissipated through the resistors? And how much through each one?

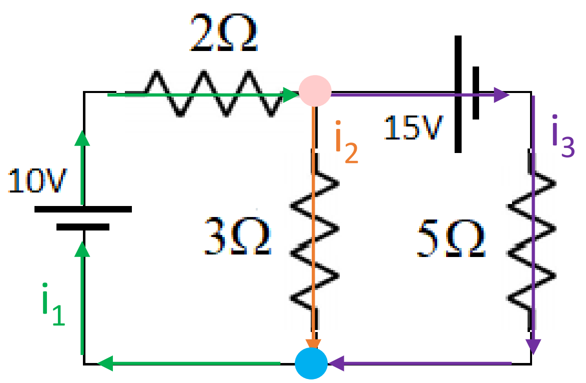


In this case we cannot combine any of the resistors in series or parallel. So we must take another route. But we've been down this route before, in the context of capacitors.



1. Label the currents in the wires in any arbitrary way:

B.3 Network of Resistors



2. At each junction, apply KCL (Kirchoff's charge law), which states that the sum of the currents leaving a junction must add up to zero.

$$\sum i's = 0 \qquad KCL$$

Applied to the red junction, we'd have:

$$-i_1 + i_2 + i_3 = 0$$

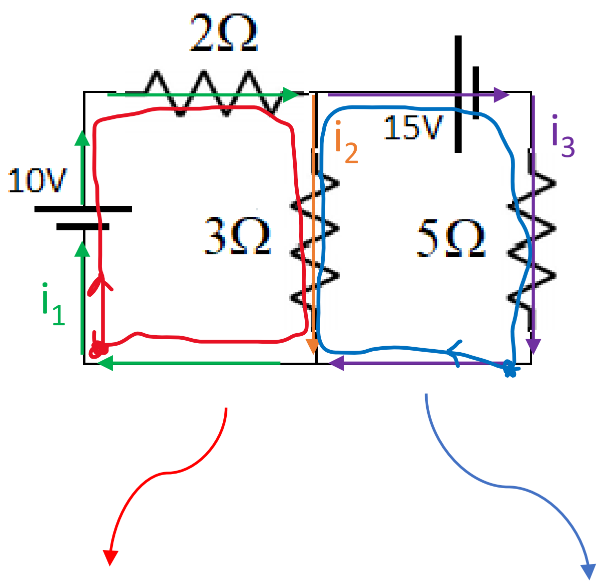
$$i_1 - i_2 - i_3 = 0$$

What about the blue junction?

$$i_1 - i_2 - i_3 = 0$$

So we needn't apply it to this junction because we'll get the same equation. Either junction works as well as the other.

B.3 Network of Resistors



$$+10-2i_1-3i_2=0$$

$$2i_1+3i_2=10$$

$$3i_2-15-5i_3=0$$

$$3i_2-5i_3=15$$

3. Apply KVL (Kirchoff's Voltage Law) to as many loops as necessary (# of unknown currents -1), which states that the sum of the potential differences around a closed loop must add up to zero.

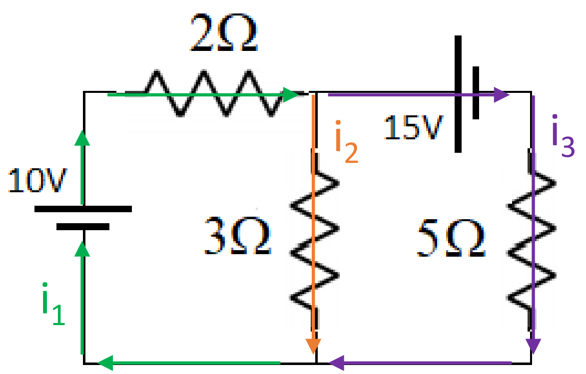
$$\sum \Delta V's=0 \qquad KVL$$

The loops you choose, their starting point, and the direction in which you circumambulate them are all arbitrary.

Important point on the resistors: the presumed direction of the electric field in wires follows the currents you drew. So when you cross a resistor in the direction of the current (electric field), the potential difference is accounted *negative*. When you cross a resistor opposite the direction of the current (electric field), the potential difference is accounted *positive*.

Important point on the batteries: the electric field in batteries has nothing to do with the currents you drew. And when you cross the battery from + terminal (the long line) to - terminal (the short line), the potential difference should be accounted *negative*. But when you cross from –terminal to +terminal, it would be accounted *positive*.

B.3 Network of Resistors



4. Solve the KCL, KVL equations for the i's

These are the three equations so far:

$$i_1 - i_2 - i_3 = 0$$
$$2i_1 + 3i_2 = 10$$
$$3i_2 - 5i_3 = 15$$

You can use a matrix to solve them if you wish. It would look like this:

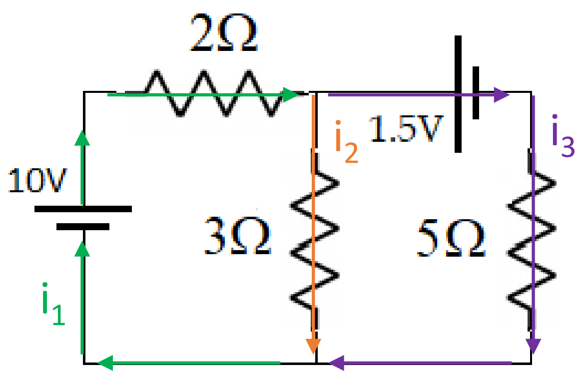
$$\begin{pmatrix} 1 & -1 & -1 \\ 2 & 3 & 0 \\ 0 & 3 & -5 \end{pmatrix}\begin{pmatrix} i_1 \\ i_2 \\ i_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 10 \\ 15 \end{pmatrix} \longrightarrow \begin{pmatrix} i_1 \\ i_2 \\ i_3 \end{pmatrix} = \begin{pmatrix} 1 & -1 & -1 \\ 2 & 3 & 0 \\ 0 & 3 & -5 \end{pmatrix}^{-1}\begin{pmatrix} 0 \\ 10 \\ 15 \end{pmatrix} \longrightarrow \begin{pmatrix} i_1 \\ i_2 \\ i_3 \end{pmatrix} = \begin{pmatrix} 1.1 \\ 2.6 \\ -1.5 \end{pmatrix}$$

The negative sign on i_3 means that we guessed the direction of the current wrong, and it actually flows the other way.

So the power delivered is: $P_{delivered} = i_1 \Delta V_1 + i_3 \Delta V_3 \quad = (1.1\text{A})(10\text{V}) + (1.5\text{A})(15\text{V}) = 34\text{W}$

And the power dissipated is: $P_{dissipated} = i_1^2 R_1 + i_2^2 R_2 + i_3^2 R_3 \quad = (1.1\text{A})^2(2\Omega) + (2.6\text{A})^2(3\Omega) + (1.5\text{A})^2(5\Omega) = 2.4\text{W} + 20.3\text{W} + 11.3\text{W} = 34\text{W}$

B.3 Network of Resistors



It's instructive to consider what would happen if we reduced the voltage of the 15V battery to 1.5V. Then our equations would be:

$$i_1 - i_2 - i_3 = 0$$
$$2i_1 + 3i_2 = 10$$
$$3i_2 - 5i_3 = 1.5$$

You can use a matrix to solve them if you wish. It would look like this:

$$\begin{pmatrix} 1 & -1 & -1 \\ 2 & 3 & 0 \\ 0 & 3 & -5 \end{pmatrix}\begin{pmatrix} i_1 \\ i_2 \\ i_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 10 \\ 1.5 \end{pmatrix} \longrightarrow \begin{pmatrix} i_1 \\ i_2 \\ i_3 \end{pmatrix} = \begin{pmatrix} 1 & -1 & -1 \\ 2 & 3 & 0 \\ 0 & 3 & -5 \end{pmatrix}^{-1}\begin{pmatrix} 0 \\ 10 \\ 1.5 \end{pmatrix} \longrightarrow \begin{pmatrix} i_1 \\ i_2 \\ i_3 \end{pmatrix} = \begin{pmatrix} 2.4 \\ 1.7 \\ 0.7 \end{pmatrix}$$

We see that current would now be flowing *against* the 1.5V battery. This means that instead of delivering power to the circuit, it would be absorbing power (essentially, recharging).

So the power delivered is: $P_{delivered} = i_1 \Delta V_1 = (2.4\text{A})(10\text{V}) = 24\text{W}$

And the power dissipated/absorbed is: $P_{dissipated} = i_1^2 R_1 + i_2^2 R_2 + i_3^2 R_3 + i_3 \Delta V_3 = (2.4\text{A})^2(2\Omega) + (1.7\text{A})^2(3\Omega) + (0.7\text{A})^2(5\Omega) + (0.7\text{A})(1.5\text{V})$

$$= 11.6\text{W} + 8.7\text{W} + 2.5\text{W} + 1.2\text{W} = 24\text{W}$$